COMBINATORICA
Akadémiai Kiadó – Springer-Verlag

COMBINATORICA 11 (1) (1991) 9–15

THE EXISTENCE OF NON-TRIVIAL HYPERFACTORIZATIONS OF K_{2n}*

ENDRE BOROS, DIETER JUNGNICKEL and SCOTT A. VANSTONE

Received May 10, 1988

A λ-*hyperfactorization* of K_{2n} is a collection of 1-factors of K_{2n} for which each pair of disjoint edges appears in precisely λ of the 1-factors. We call a λ-hyperfactorization *trivial* if it contains each 1-factor of K_{2n} with the same multiplicity γ (then $\lambda = \gamma(2n-5)!!$). A λ-hyperfactorization is called *simple* if each 1-factor of K_{2n} appears at most once. Prior to this paper, the only known non-trivial λ-hyperfactorizations had one of the following parameters (or were multipliers of such an example)

(i) $2n = 2^a + 2$, $\lambda = 1$ (for all $a \geq 3$); cf. Cameron [3];

(ii) $2n = 12$, $\lambda = 15$ or $2n = 24$, $\lambda = 495$; cf. Jungnickel and Vanstone [8].

In the present paper we show the existence of non-trivial simple λ-hyperfactorizations of K_{2n} for all $n \geq 5$.

1. Introduction

1-factorizations (and more generally, c-factorizations) of the complete graph K_{2n} are among the best-known objects in graph theory; see Bondy and Murty [2] for terms from graph theory and the survey paper Mendelsohn and Rosa [11] for 1-factorizations. In this paper, we shall be concerned with the stronger notion of a λ-*hyperfactorization.* This is a collection $\mathbf{H}$ of 1-factors of K_{2n} satisfying the following condition:

(HF) Any two disjoint edges of K_{2n} belong to exactly λ 1-factors in $\mathbf{H}$.

Note that we allow the use of 1-factors more than once; if $\mathbf{H}$ contains each 1-factor at most once, we shall call $\mathbf{H}$ *simple.* The integer λ is also called the *index* of the *hyperfactorization* $\mathbf{H}$. Before we discuss examples, let us first note the following simple observations.

Lemma 1.1. *Let* $\mathbf{H}$ *be a* λ*-hyperfactorization of* K_{2n}. *Then* $\mathbf{H}$ *consists of* $\lambda(2n-1)(2n-3)$ *1-factors, and each edge of* K_{2n} *is in exactly* $\lambda(2n-3)$ *of these 1-factors.*

Lemma 1.2. *The total number of 1-factors of* K_{2n} *is* $(2n-1)!! = (2n-1) \times (2n-3) \cdot \ldots \cdot 3 \cdot 1$. *Thus all 1-factors of* K_{2n} *form a hyperfactorization of index* $\lambda = (2n-5)!!$.

The proofs of 1.1 and 1.2 are left to the reader. The examples provided by 1.2 are, of course, trivial; we will call each hyperfactorization containing every 1-factor of K_{2n} (with constant multiplicity) *trivial.* Our interest in this paper will be in the existence question for non-trivial simple hyperfactorizations, a not entirely

AMS subject classification (1980): 05 B 30 (primary), 05 C 99 (secondary)

*This is the revised version of an earlier preprint bearing the same title (Research Report CORR 87–51, Dept. of Combinatorics and Optimization, University of Waterloo).

trivial problem, as we shall presently see from indicating the known results. The only known infinite series of examples (up to taking multiples) is as follows.

Result 1.3. *(Cameron [3]). There exists a hyperfactorization in K_{2^a+2} for all $a \geq 2$. More generally, if there exists a finite projective plane of even order n which contains a hyperoval then there exists a hyperfactorization of index 1 in K_{n+2}.*

The examples mentioned in 1.3 cover all values of n for which a 1-hyperfactorization in K_{2n} is known to exist. Non-existence results are even scarcer. Cameron [3] and Mathon [10] show the non-existence of a 1-hyperfactorization in K_8, and Lam, Thiel, Swiercz and McKay [9] have ruled out the case K_{12} by a computer search. It should be noted that 1-hyperfactorizations do not necessarily belong to hyperovals in projective planes of order n, as an example of Mathon [10] (for K_{10}) shows. They do, however, always admit a nice geometric interpretation as a "partial geometry", (see Mathon [10] and, for arbitrary λ, Jungnickel and Vanstone [8]).

The only other known nontrivial hyperfactorizations were constructed by two of the present authors in [8], using the Mathieu groups M_{12} and M_{24} (see, e.g., [1] for a combinatorial treatment of these groups):

Proposition 1.4. *There exists both a 15-hyperfactorization in K_{12} and a 495-hyperfactorization in K_{24}.*

In the present paper, we shall show the existence of non-trivial hyperfactorizations for all possible values of n.

Main theorem. *There exists a non-trivial simple hyperfactorization in K_{2n} for all $n \geq 5$.*

We shall prove this result by means of a recursive construction in Section 2. This leaves the difficult problem of determining (for given n) the spectrum of those λ for which a (simple) λ-hyperfactorization of K_{2n} exists. We shall obtain some further series of examples in Section 3.

Chris Godsil has noticed that the existence of nontrivial hyperfactorizations of K_{2n} (for $n \geq 5$) can be obtained as a special case of a general theory introduced in his paper [4]. However, his approach gives no control over the size of λ and will, in particular, not guarantee the existence of *simple* nontrivial λ-hyperfactorizations.

We conclude the introduction by pointing out that the existence of hyperfactorizations of small index is of considerable interest in Design Theory. It was shown in [6] that any λ-hyperfactorization of K_{2n} yields a 5-design $S_{15\lambda}(5,6,2n)$. We refer the reader to Beth, Jungnickel and Lenz [1] for background on designs. Result 1.3 thus yields the only known series of 5-designs with a fixed λ-value, i.e., $S_{15}(5,6,2^a+2)$ for $a \geq 3$. Unfortunately, the hyperfactorizations constructed in the present paper are of no interest for design theory, as they result in designs $S_{15(2n-5)c}(5,6,2n)$ (for suitable values of c), whereas the trivial 5-design formed by all 6-subsets of a $2n$-set is an $S_{2n-5}(5,6,2n)$. Thus our examples only yield (quasi-)multiples of trivial 5-designs.

2. A recursive construction

In this section we shall prove out principal result, i.e. the existence of simple non-trivial hyperfactorizations of K_{2n} for all $n \geq 5$. This will be a consequence of the following recursive construction.

Theorem 2.1. *Assume the existence of a (simple) λ-hyperfactorization in K_{2n}. Then there also exists a (simple) γ-hyperfactorization in K_{2n+2}, where $\gamma = \lambda(2n-3)$.*

Proof. Let $X = \{1, 2, \ldots, 2n+2\}$ and let $\mathbf{F}$ be a hyperfactorization on $\{3, 4, \ldots, 2n+2\} = Y$. We put $e = \{1, 2\}$ and from the following sets of 1-factors on X:

$$\mathbf{H}_e = \{F \cup \{e\} \ : \ F \in \mathbf{F}\}$$

and, for each ordered pair (a, b) of distinct elements a, b of Y,

$$\mathbf{H}_{ab} = \{(F \setminus \{\{a,b\}\}) \cup \{\{1,a\},\{2,b\}\} \ : \ F \in \mathbf{F}, \{a,b\} \in F\}.$$

We claim that

$$\mathbf{H} = \mathbf{H}_e \cup \bigcup_{(a,b)} \mathbf{H}_{ab}$$

is the desired γ-hyperfactorization on X. Thus consider any pair of disjoint edges $\{a, b\}$, $\{c, d\}$. Then we have one of the following four cases.

(i) $\{c, d\} = \{1, 2\} = e$ and a, $b \in Y$. Since no $\mathbf{H}_{ab}$ contains the edge e, we obtain exactly $\lambda(2n-3)$ 1-factors in $\mathbf{H}$ containing both edges. These are the 1-factors determined by the $\lambda(2n-3)$ 1-factors in $\mathbf{F}$ containing the edge $\{a, b\}$.

(ii) Exactly 3 of the elements a, b, c, d are in Y, say $d = 1$ (the case $d = 2$ is similar) and a, b, $c \in Y$. Then a 1-factor in $\mathbf{H}$ containing both edges has to belong to $\mathbf{H}_{cy}$ for some $y \in Y$. Now there are exactly $\lambda(2n-3)$ 1-factors $F \in \mathbf{F}$ containing the edge $\{a, b\}$. Each of these contains a unique edge of the form $\{c, y\}$. Thus we again obtain exactly $\lambda(2n-3)$ 1-factors in $\mathbf{H}$ which contain both edges.

(iii) $\{a, b\} = \{1, b\}$ and $\{c, d\} = \{2, d\}$. The only 1-factors in $\mathbf{H}$ containing both edges belong to $\mathbf{H}_{bd}$. Since the edge $\{b, d\}$ is in exactly $\lambda(2n-3)$ 1-factors in $\mathbf{F}$, we again obtain $\lambda(2n-3)$ 1-factors $H \in \mathbf{H}$ containing both edges.

(iv) a, b, c, $d \in Y$. Then $\{a, b\}$ and $\{c, d\}$ are in exactly λ common 1-factors $F \in \mathbf{F}$. For each such F we obtain $2n-3$ 1-factors $H \in \mathbf{H}$ containing both edges: one 1-factor $F \cup \{e\}$ in $\mathbf{F}_e$ and one 1-factor each in $\mathbf{H}_{xy}$ and $\mathbf{H}_{yx}$ for each of the remaining $n-2$ edges $\{x, y\} \in F$. Again, the total number of 1-factors in $\mathbf{H}$ containing both edges is $\lambda(2n-3)$.

Thus $\mathbf{H}$ is indeed a γ-hyperfactorization on X. It is obvious from our construction that $\mathbf{H}$ is simple if and only if $\mathbf{F}$ is simple. ∎

Since there exists a (simple) 1-hyperfactorization in K_{10} (see Result 1.3), a recursive application of Theorem 2.1 suffices to prove our principal result:

Main Theorem. *There exists a simple non-trivial hyperfactorization of K_{2n} for all $n \geq 5$.*

Remarks.

Clearly no non-trivial hyperfactorization can exist in K_{2n} for $n \leq 3$. For $n = 4$, the trivial simple hyperfactorization has index $\lambda = 3$. Since it is known that K_8 does not admit a 1-hyperfactorization (see [3] or [10]), we cannot have a simple 2-hyperfactorization of K_8 either (for its complement would be a 1-hyperfactorization). Thus Theorem 2.2 does not hold for $n = 4$. Nevertheless, Joe Horton [6] has recently constructed a 2-hyperfactorization of K_8. His example contains exactly 7 repeated 1-factors and is invariant under the cyclic group of order 7.

Example 2.4. Using Theorem 2.1 and Result 1.3, we in particular obtain a simple λ-hyperfactorization for $\lambda = 2^a - 1$ in K_{2^a+4} $(a \geq 3)$. Thus we get a simple 7-hyperfactorization in K_{12}, whereas the previously known example has $\lambda = 15$ (see Proposition 1.4).

In view of Theorem 2.2, we now have the problem of determining the spectrum of those λ for which a simple λ-hyperfactorization in K_{2n} exists. We first observe the following:

Lemma 2.5. *The existence of a simple λ-hyperfactorization of K_{2n} implies that of a simple μ-hyperfactorization where $\mu = (2n-5)!! - \lambda$.*

Proof. Take the complement of the λ-hyperfactorization (with respect to the set of all 1-factors of K_{2n}). ∎

Thus we may restrict ourselves to cases with $\lambda < (2n-5)!!/2$. Using the known 1-hyperfactorizations, see Theorem 1.3, we get the following results.

Theorem 2.6. *There exists a simple λ_a-hyperfactorization of K_{2n} $(n \geq 5)$, where*

$$\lambda_a = (2n-5)\dots(2^a - 1)$$

for all $a \geq 3$ satisfying $2^{a-1} < n - 1$. When $2^{a-1} = n - 1$ then $\lambda_a = 1$.

Proof. By assumption, $2^a + 2 \leq 2n$. For $2^a + 2 < 2n$ the assertion follows by recursively applying Theorem 2.1 to a 1-hyperfactorization in K_{2^a+2}. ∎

Corollary 2.7. *For $n \geq 5$, there exists a simple non-trivial λ-hyperfactorization of K_{2n} with*

$$\lambda \leq (2n-5)\dots(n-1)$$

Proof. Selecting the largest possible value for a in Theorem 2.6 guarantees that also $n \leq 2^a$. ∎

We shall construct further examples of simple hyperfactorizations in the next section.

3. Unions of simple hyperfactorizations

In this section we shall give a construction that allows (under certain conditions) to produce a simple $(\lambda + \mu)$-hyperfactorization in K_{2n} from simple λ- and μ-hyperfactorizations. The method of proof is somewhat similar to an argument used by van Trung [12] for a construction of simple t-designs.

Theorem 3.1. *Assume the existence of simple λ- and μ-hyperfactorizations in K_{2n} respectively. If*

$$(*) \qquad \lambda\mu(2n-1)(2n-3)2^{n-2} < (2n-4)(2n-5)\dots(n-1),$$

then there also exists a simple $(\lambda+\mu)$-hyperfactorization in K_{2n}.

Proof. Let $\mathbf{F}$ and $\mathbf{G}$ be simple hyperfactorizations of K_{2n} with indices λ and μ, respectively. Then

$$\mathbf{H}^\sigma = \mathbf{F} \cup \mathbf{G}^\sigma$$

is a $(\lambda+\mu)$-hyperfactorization for any permutation σ in the symmetric group S_{2n} on the vertex set $\{1,\dots,2n\}$ of K_{2n}. (Here unions are to be understood as multisets, with multiplicities taken into account.) We want to show that there is a choice for σ leading to a simple $\mathbf{H}^\sigma$ provided that $(*)$ is satisfied.

Given any two 1-factors F and F' of K_{2n}, there are exactly $2^n n!$ permutations $\sigma \in S_{2n}$ with $F^\sigma = F'$. Thus, given $F \in \mathbf{F}$ and $G \in \mathbf{G}$, there will be exactly $2^n n!$ choices for σ such that $G^\sigma = F$. Then $\mathbf{H}^\sigma$ contains F as a repeated 1-factor. Since there are $\mu(2n-1)(2n-3)$ choices for G and $\lambda(2n-1)(2n-3)$ choices for F, at most $\lambda\mu(2n-1)^2(2n-3)^2 2^n n!$ permutations $\sigma \in S_{2n}$ lead to a nonsimple $\mathbf{H}^\sigma$. Thus we can clearly choose a σ for which $\mathbf{H}^\sigma$ is simple if $\lambda\mu(2n-1)^2(2n-3)^2 2^n n! < (2n)!$ which is equivalent to $(*)$. ∎

Corollary 3.2. *Assume the existence of a 1-hyperfactorization of K_{2n}. Then there exists a simple γ-hyperfactorization of K_{2n} provided that*

$$(**) \qquad (\gamma-1)(2n-1)(2n-3)^{n-2} < (2n-4)(2n-5)\dots(n-1).$$

Proof. Apply 3.1 recursively for $\lambda = 1,\dots,\gamma-1$ with $\mu = 1$. ∎

We shall now use our results to discuss the spectrum of those λ for which a (simple) λ-hyperfactorization is known ti exist for $2n \le 24$. This updates the discussion in [8]. In view of 2.5, we restrict ourselves to $\lambda < (2n-5)!!/2$ in the simple case. The results exhibited are easily obtained from 1.3, 1.4, 2.1 and 3.2. We will still list them in tabular form, leaving the details to the reader.

$2n$	trivial λ: $(2n-5)!!$	simple examples for indices	not necessarily simple examples for indices
6	1	1	
8	3	3	2
10	15	1	$\lambda \ge 2$
12	105	7;15	$14, 21, 22, 28, 29, 30, \dots, \lambda \ge 84$
14	945	63;135	$9x$ (x as for $2n = 12$)
16	10,395	693;1485	$99x$ (x as for $2n = 12$)
18	135,135	$\lambda \le 530$;9009;19305	$\lambda \ge 531$
20	2,027,025	$15x$ ($x \le 530$);135,135;289,575	$15x$ ($x \ge 531$)
22	34,459,425	$255x$ ($x \le 530$);2,297,295;4,922,775	$255x$ ($x \ge 531$)
24	654,729,075	$495y$ ($y \le 6$);$4845x$ ($x \le 530$); 43,648,605;93,532,725	$4845x + 495y$

4. Conclusion

Prior to this paper, non-trivial hyperfactorizations of K_{2n} were known to exist only for $2n = 12$, 24 or $2^a + 2$ ($a \geq 3$). We have substantially improved this by showing the existence of simple non-trivial hyperfactorizations in K_{2n} for all $n \geq 5$. However, the problem of determining the spectrum of all λ for which a simple λ-hyperfactorization of K_{2n} exists, remains largely unsolved. In this connection, we feel that the following specific problems are important.

Problem 1. *Is there a recursive construction that produces a simple γ-hyperfactorization from a simple λ-hyperfactorization where $\gamma < \lambda(2n-3)$?*

Problem 2. *Find new direct constructions for simple hyperfactorizations.*

Problem 3. *Does there exist a 1-hyperfactorization for any K_{2n} with $2n \neq 2^a + 2$? Find non-existence results for 1-hyperfactorizations. Does the existence of a 1-hyperfactorization of K_{2n} imply the existence of a projective plane of order $2n-2$ containing a hyperoval, if n is sufficiently large? (This is not true for $2n = 8$ by the example of Mathon [10], as mentioned in the introduction.)*

Problem 4. *Is the spectrum of λ for which a simple λ-hyperfactorization of K_{2n} exists an interval (and thus of the form $[c, \ldots, (2n-5)!! - c]$ by Lemma 2.5)?*

Acknowledgement. The first author is on leave from the Computer and Automation Institute of the Hungarian Academy of Sciences; he gratefully acknowledges the support by NSF grant #ECS85-03212 and by AFOSR grant #0271. During the time of this research, the second author was a Visiting Professor at the University of Waterloo. He would like to thank this institution for its hospitality and NSERC for financial support under grant #I-0367. The last author gratefully acknowledges the partial support by NSERC under grant #A 9258.

References

[1] T. BETH, D. JUNGNICKEL, and H. LENZ: *Design Theory*, Bibliographisches Institut, Mannheim (1985), and Cambridge University Press, Cambridge (1986).

[2] J. A. BONDY, AND U. S. R. MURTY: *Graph Theory with applications*, North Holland, New York (1976).

[3] P. J. CAMERON: *Parallelisms of complete designs.* Cambridge University Press, Cambridge (1976).

[4] C. GODSIL: Polynomial spaces. *Discrete Math.*, **73** (1988/89), 71–88.

[5] J. W. P. HIRSCHFELD: *Projective geometries over finite fields.* Oxford University Press, Oxford (1979).

[6] D. J. HORTON: Personal communication.

[7] D. R. HUGHES, and F. C. PIPER: *Projective planes.* Springer, Berlin–Heidelberg–New York (1973).

[8] D. JUNGNICKEL, and S. A. VANSTONE: Hyperfactorizations of graphs and 5-designs. *J. Kuwait Univ.*, (Sci.) **14** (1987), 213–224.

[9] C. W. H. LAM, L. THIEL, S. SWIERCZ, and J. MCKAY: The nonexistence of ovals in a projective plane of order 10, *Discrete Math.* **45** (1983), 318–321.

[10] R. MATHON: The partial geometries $pg(5,7,3)$. *Congressus Numerantium*, **31** (1981), 129–139.

[11] E. MENDELSOHN, and A. ROSA: One-factorization of the complete graph — a survey. *J. Graph Th.* **9** (1985), 43–65.

[12] T. VAN TRUNG: The existence of an infinite family of simple 5-designs. *Math. Z.* **187** (1984), 285–287.

Endre Boros

RUTCOR, Hill Center for the Mathematical Sciences
Rutgers Univ., New Brunswick, NJ 08903, U.S.A.

Dieter Jungnickel

Mathematisches Institut,
Justus-Liebig - Universität Giesen
Arndstr. 2, D-6300 Giessen, F. R. Germany

Scott A. Vanstone

Department of Combinatorics
and Optimization
University of Waterloo,
Waterloo, Ontario N2L 3G1, Canada